Math 181

Applications of Integration
Arc Length

Let C be a curve given parametrically by the equations

$$x = f(t) \quad y = g(t) \quad a \le t \le b$$

We assume the functions *f* and *g* have continuous first derivatives on the interval [a, b] that are not simultaneously zero. Such functions are said to be **continuously differentiable**, and the curve C defined by them is called a **smooth curve**.

If a curve C is defined parametrically where *f* and *g* are continuously differentiable and C is traversed exactly once as *t* increases from *t* = *a* to *t* = *b*, the **length of C** is the definite integral

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$$

A smooth curve C does not double back or reverse the direction of motion over the time interval [a, b] since $(f')^2 + (g')^2 > 0$ throughout the interval.

Example: Find the length of the curve $x = \frac{t^2}{2} \quad y = \frac{(2t+1)^{\frac{3}{2}}}{3} \quad 0 \le t \le 4$

Given a continuously differentiable function $y = f(x), a \le t \le b$, we can assign $x = t$ as a parameter. The graph of the function *f* is then the curve C defined parametrically by

$$x = t \quad y = f(t) \quad a \le t \le b$$

Then

$$\frac{dx}{dt} = 1 \quad \frac{dy}{dt} = f'(t)$$
$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 1 + [f'(t)]^2 = 1 + [f'(x)]^2$$

If *f* is continuously differentiable on the closed interval [a, b], the length of the curve is

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

Example: Find the length of the curve $x = \frac{y^{\frac{3}{2}}}{3} - y^{\frac{1}{2}} \quad from \quad y = 1 \; to \; y = 9$

Example: Find a curve through the point (0, 1) whose length integral is

$$L = \int_1^2 \sqrt{1 + \frac{1}{y^4}}\, dy$$

How many such curves are there? Give reasons for your answer.