Math 181

Applications of Integration
Centroids and Centers of Mass

According to the law of the lever, two masses m_1 and m_2 on opposite sides and at respective distances d_1 and d_2 from the fulcrum of a lever will balance provided that $m_1 d_1 = m_2 d_2$. Think of the x-axis as the location of a (weightless) lever arm supporting various point masses and of the origin as a fulcrum. Then a more general form of the law of the lever states that (particle) masses $m_0, m_1, m_2, \ldots m_n$ with respective coordinates $x_0, x_1, x_2, \ldots x_n$ will balance provided that

$$\sum_{i=0}^{n} m_i x_i = 0$$

Now consider arbitrary masses $m_1, m_2, \ldots m_n$ at the points $x_1, x_2, \ldots x_n$. Then a single particle with mass

$$m = \sum_{i=1}^{n} m_i$$

at position $-\bar{x}$ will balance these n masses proved that

$$-m\bar{x} + \sum_{i=1}^{n} m_i x_i = 0;$$

that is, provided that

$$\bar{x} = \frac{1}{m} \sum_{i=1}^{n} m_i x_i$$

The point $\bar{x}$ is called the **center of mass**, and the sum $\sum_{i=1}^{n} m_i x_i$ is called **the moment** of the system of masses about the origin.

Now consider a system of n particles with masses $m_1, m_2, \ldots m_n$ located in the plane at the points with respective coordinates $(x_1, y_1), (x_2, y_2), \cdots (x_n, y_n)$. In analogy with the one-dimensional case just discussed, we define **moment M_y** of this system of masses **about the y-axis** and its **moment M_x about the x-axis** by means of the equations

$$M_y = \sum_{i=1}^{n} m_i x_i \text{ and } M_x = \sum_{i=1}^{n} m_i y_i$$

The center of mass of this system of *n* particles is the point $(\bar{x}, \bar{y})$ with coordinates defined to be

$$\bar{x} = \frac{M_y}{m} \quad \bar{y} = \frac{M_x}{m}$$

where $m = \sum_{i=1}^{n} m_i$.

As the number of particles we consider increases while their masses decrease in proportion, their aggregate more and more closely resembles a plane region of varying density. Next we consider a flat plat (called a lamina) with uniform density ρ that occupies a region $\mathcal{R}$ of the plane. We wish to locate the center of mass of the plate, which is called the **centroid** of $\mathcal{R}$.

The **moment about the x-axis** and the **moment about the y-axis** are defined by

$$M_x = \int_a^b \rho(x) \cdot \left[\frac{f(x) + g(x)}{2} \right] \cdot [f(x) - g(x)] dx$$

$$M_y = \int_a^b \rho(x) \cdot x \cdot [f(x) - g(x)] dx$$

where $\rho(x)$ is the density function.

The total mass is defined by

$$m = \int_a^b \rho(x) \cdot [f(x) - g(x)] dx$$

Then the center of mass is the point $(\bar{x}, \bar{y})$ given by

$$\bar{x} = \frac{M_y}{m} \quad \bar{y} = \frac{M_x}{m}$$

Example: Find the centroid of the region bounded by the line $y = x$ and the parabola $y = x^2$ with constant density.