Math 100

Mathematical Models

When we attempt to formulate the solution of a physical problem by discovering equations and graphs that describe the problem, we refer to this process as **mathematical modeling**. A **mathematical model** of a problem enables one to understand and to quantify what is happening at this time, as well as to predict future outcomes and to explain past events. Mathematical models can be incredibly powerful tools; however, as we shall see, there are always questions concerning the validity of the model that must be considered.

$$m = \frac{\Delta y}{\Delta x}$$

$$y = mx + b$$

$$y - y_1 = m(x - x_1)$$

Example 1: You plan to rent a truck in Canada. The local *Trucks-R-Us* rental company charges $39 (U. S. dollars) per day plus 16¢ per kilometer. Set up a linear model that will describe a one-day rental both algebraically and geometrically. What would be the cost of renting the truck if you expect to drive 100 kilometers? How many kilometers could you drive if you wanted to keep your total costs under $200?

Solution: No matter how many kilometers you drive, you must pay the $39. This $39 represents the fixed cost of the problem. Then you need to pay 16¢ per kilometer in addition. Recall from algebra that slope is a rate of change. You need to pay 16¢ **per kilometer** (a rate of change). The 16¢ represents the slope. Therefore, the algebraic model for the above problem is

$$C(x) = .16x + 39$$

$$x = \#\ of\ kilometers\ driven$$

$$C(x) = \cos t\ of\ renting\ the\ truck$$

Next, let's graph this model (a geometric interpretation of the problem).

A Model for Developing a Mathematical Model

1. Choose data that appear to be related.
2. Set up a coordinate system, label axes, and choose appropriate scales.
3. Plot the data points as ordered pairs on the coordinate system.
4. Sketch a **curve** that passes through the points.
5. Determine the equation of the curve.
6. Consider the reasonableness of your results.
7. Consider any limitations to your model.
8. Consider the appropriateness of the scales.

Example 2: Still planning your move, you call Moving-Mania to see if you can get a better deal. They charge $25 per day plus 24¢ per kilometer. Which company offers a better deal?

Example 3: Carol is starting a running program and wants to monitor her pulse as she progresses. She notes that after running for 30 minutes her pulse is 124 and after 40 minutes it is 136. *If the model is linear*, what would her pulse be after 60 minutes? Discuss your analysis.

Homework

1. Susan needs to buy a new hot water heater and has narrowed her choice down to two different models. One costs $525 and 8¢ per hour to operate. The other costs $700 but only 5¢ per hour to operate.
    a) Construct a model to compare the costs of each hot water heater using equations and graphs.
    b) Explain which model would be a better buy.

2. Jan needs to purchase a new copier for her department and has two choices. The first machine costs $20,000 and 2¢ per copy. The other machine costs $17,000 and 2.5¢ per copy.
    a) Construct a model using equations and graphs to compare the costs of each copier.
    b) Explain which copier would be a better buy.

3. The population of Smallville in 1990 was 12,000. If the population has been growing at a rate of 8% per year since then:
    a) Set up a model using equations and graphs to predict the future population.
    b) What does your model predict the population will be in the year 2000?
    c) Explain any limitations of the model.

4. As a craftsman making feather dusters, you have invested $250 in materials. You expect to sell your dusters for $7.50 each.
    a) Set up a model, draw a graph, and determine an equation to calculate your profit.
    b) Use the graph to estimate how many dusters you would have to sell to break even.
    c) Use the equation to determine how many dusters you would have to sell to break even.
    d) How many dusters would you have to sell to earn a profit of at least $100?

Quadratic models

Example 1: You want to construct a rectangular tabletop so as to make the most of your materials. You have just enough cherry wood to edge the table if the perimeter of the top is exactly 22 feet. What must the dimensions be in order to make the area of the top as large as possible?

One way to approach this problem is to make a table.

Length	Width	Area

Another way to approach the problem is algebraically.
The perimeter is 22 feet.

$$2L + 2W = 22$$

$$L = 11 - W$$

The area of a rectangle is Length times Width.

$$A = L \cdot W$$

$$= (11 - W)W$$

$$= -W^2 + 11W$$

Recall from algrebra that the graph of this equation is a parabola. The maximum or minimum will occur at the vertex. The x-coordinate of the vertex of a parabola is $x = -\frac{b}{a}$ if the quadratic equation is $y = ax^2 + bx + c$.

Therefore, the value of W at the vertex is $W = \frac{-b}{2a} = -\frac{11}{(-2)} = 5.5$.

Example 2: Suppose you want to fence in a rectangular plot of land along a river bank using 3,000 yards of barbed wire. You do not need to fence the side along the river bank which acts as a natural barrier. What are the dimensions of the plot of land that encloses the largest area?

Homework

1. Suppose that 600 yards of fencing material are available to fence in two adjacent rectangular corrals sharing a common fence. Find the dimensions that will enclose the maximum area.
2. You want to build a rectangular enclosure. Materials along the sides running east-west cost $7 per foot, while materials along the sides running north-south cost $10 per foot. If you have $4800 to spend, find the dimensions of the rectangle that encloses the largest area.
3. The height of an object thrown vertically upward is given by $h(t) = -16t^2 + 96t + 256$.
    a) Graph the equation.
    b) Find the h-intercept. How would you interpret that point?
    c) Find the t-intercepts. How would you interpret those points?
    d) How long is the object in the air?
    e) Show that there are two different times when the height of the object is 256 feet. Explain why that is the case.

Exponential Models

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$A = amount\ accumulated\ after\ t\ years$

$P = initial\ amount\ invested$

$r = \text{int}\,erest\ rate$

$n = \#\ of\ compoundings\ per\ year$

Example 1: Suppose you wish to invest $1000 and leave it in the account for ten years. You approach two banks, which offer the following options:
Bank 1: r = 6.25%, compounded quarterly
Bank 2: r = 6%, compounded daily
Which bank offers you a better return on your investment?

Homework

1. How long will it take for an investment of $10,000 invested at an interest rate of 8.75% to double if the money is
   a) Compounded annually.
   b) Compounded quarterly.
   c) Compounded daily.

2. Planning for your new child's education, you decide to take the gift from grandma of $1000 and invest it in an account that pays 6.5% compounded daily.
   a) If you leave the money and the accumulated interest in the account for 18 years, what will its value be at that time?
   b) How long would you have to leave the money in the account for it to grow to a value of $10,000?

3. The world's overall population growth rate is 1.9% per year.
   a) How long will it take for the population of the world to double?
   b) How long will it take for the population of the world to triple?
   c) How long will it take for the population of the world to increase tenfold?
   d) If the rate of growth in the United States were about the same as the overall world population growth rate, how long will it take for the population of the United States to double?
   e) Discuss the implications of this model.

$$P = P_0(1+r)^t$$

$$P_0 = initial\ \ population$$

4. If Superman were trapped in a room by the arch-villain, Lex Luther, and an initial sample of krypton-91 of 5000 grams.
   a) How much of the sample would be left after 1 minute?
   b) How much of the sample would be left after 5 minutes?
   c) How long would it take for the initial sample to be reduced to 1 gram?

$$N = N_0(.5)^{t/h}$$

$$N_0 = initial\ \ amount$$

$$h = half - life$$

$$half - life\ of\ \ krypton - 91 = 10\ \text{sec}\, onds$$